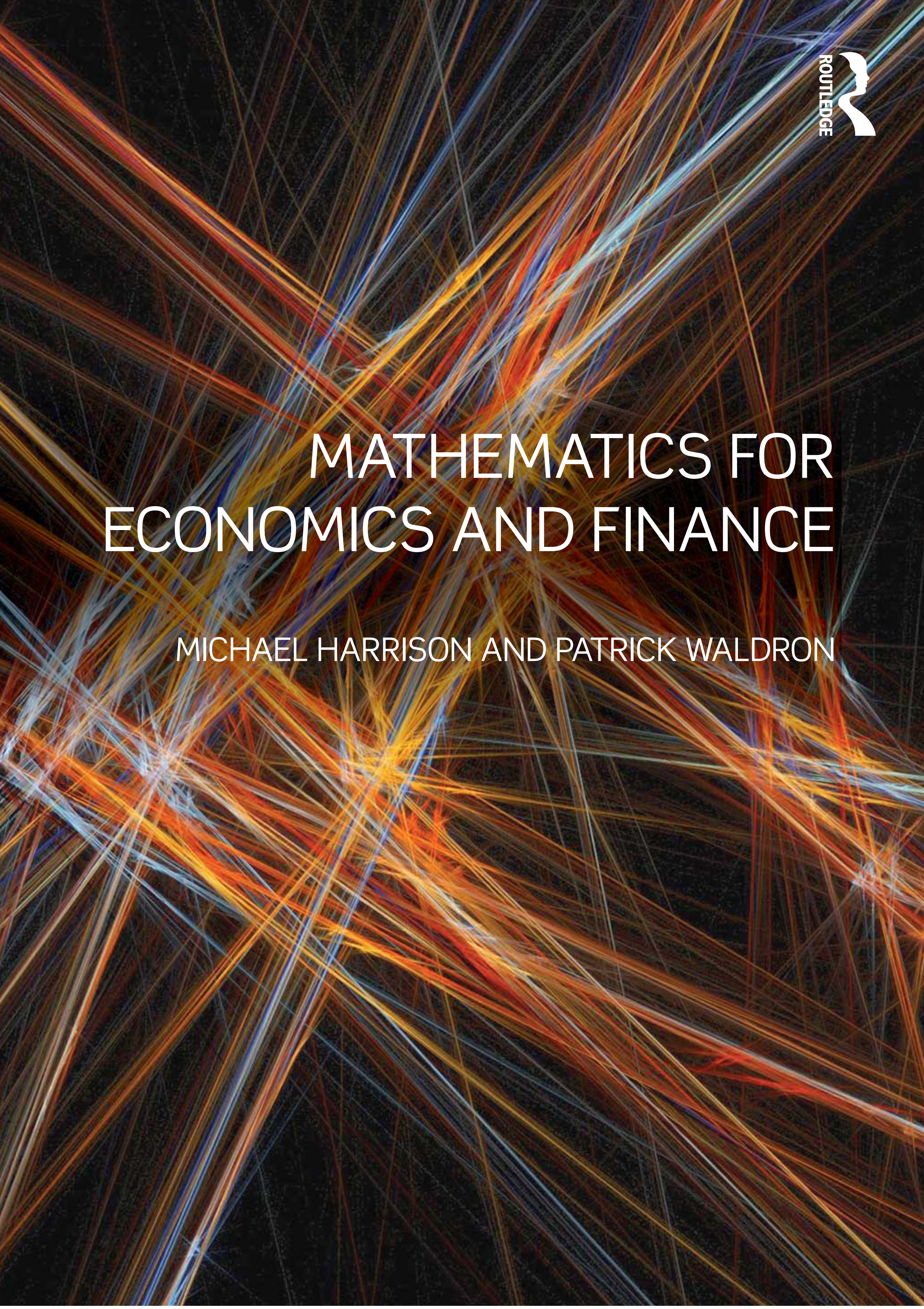
ROUTLEDGE
MATHEMATICS FOR
ECONOMICS AND FINANCE
MICHAEL HARRISON AND PATRICK WALDRON

Mathematics for Economics and Finance

The aim of this book is to bring students of economics and finance who have only an introductory background in mathematics up to a quite advanced level in the subject, thus preparing them for the core mathematical demands of econometrics, economic theory, quantitative finance and mathematical economics, which they are likely to encounter in their final-year courses and beyond. The level of the book will also be useful for those embarking on the first year of their graduate studies in Business, Economics or Finance.

The book also serves as an introduction to quantitative economics and finance for mathematics students at undergraduate level and above. In recent years, mathematics graduates have been increasingly expected to have skills in practical subjects such as economics and finance, just as economics graduates have been expected to have an increasingly strong grounding in mathematics.

The authors avoid the pitfalls of many texts that become too theoretical. The use of mathematical methods in the real world is never lost sight of and quantitative analysis is brought to bear on a variety of topics including foreign exchange rates and other macro level issues. This makes for a comprehensive volume which should be particularly useful for advanced undergraduates, for postgraduates interested in quantitative economics and finance, and for practitioners in these fields.

Michael Harrison is Emeritus Senior Lecturer and Fellow of Trinity College Dublin, where he lectured from 1969 to 2009. He currently lectures in the School of Economics at University College Dublin.

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Foreword

This book has two parts, the first labelled MATHEMATICS, the second APPLICATIONS. Specifically, the applications are to economics and finance. In their Preface, the authors, Michael Harrison and Patrick Waldron, advise that the book is written for "advanced undergraduates of either mathematics or economics". However, the advanced undergraduate in economics who does not have a substantial exposure to mathematics will find Part I of this book challenging. The advanced undergraduate in mathematics, on the other hand, should find this book quite suitable for: reviewing branches of mathematics used in theoretical economics and finance; learning some aspects of these branches of mathematics, perhaps not covered in their mathematics courses, but useful in economic and financial applications; and then being presented with a comprehensive survey of economic and financial theory that draws upon this mathematics.

The advanced undergraduate in economics can use Part I as a checklist as to what mathematical background is expected in the applications in Part II. If the student has enough background to work through the presentation of the subject, then Part I is sufficient in that area. But as the authors note, their "development is often rather rapid". If the student finds the material too compact in an area, he or she can seek a suitable alternative text, which explains the subject less rapidly, then return to the Harrison and Waldron presentation.

In general, then, the two parts of this volume represent, first, an orderly and comprehensive assemblage of the mathematics needed for a great deal of important economic and financial theory; and, second, a presentation of the theory itself. The diligent student who makes his or her way through this volume will have gained a great deal of mathematical power, and knowledge of much economic and financial theory.

Any volume must have its inclusions and exclusions. One topic that has quite limited space in this volume is optimization subject to inequality constraints, such as the problem of tracing out mean–variance efficient sets subject to any system of linear equality, and/or (weak) inequality constraints in which some or all variables may be required to be non-negative. Markowitz and Todd (2000) is devoted to this topic. If the reader decides to read the latter after reading the present book, he or she will find that the present book's extensive coverage of vectors and matrices will be a big help. More generally, the mathematics background presented here will allow the student to move comfortably in the area of "computational finance" as well as financial theory.

Harry M. Markowitz
San Diego, California
October 2010

Preface

This book provides a course in intermediate mathematics for students of economics and finance. It originates from a series of lectures given to third-year undergraduates in the Faculty of Business, Economic and Social Studies and the School of Mathematics at the University of Dublin, Trinity College. The prime aim is to prepare students for some of the mathematical demands of courses in econometrics, economic theory, quantitative finance and mathematical economics, which they may encounter in their final-year programmes. The presentation may also be useful to those embarking on the first year of their postgraduate studies. In addition, it serves as an introduction to economics and finance for mathematics students.

In recent decades, mathematics graduates have been increasingly expected to have skills in practical subjects such as economics and finance, while economics graduates have been expected to have an increasingly strong grounding in mathematics. The growing need for those working in economics and finance to have a strong foundation in mathematics has been highlighted by such layman's texts as Ridley (1993), Kelly (1994), Davidson (1996), Bass (1999), Poundstone (2005) and Bernstein (2007). The present book is, in part, a response to these trends, offering advanced undergraduates of either mathematics or economics the opportunity to branch into the other subject.

There are many good texts on mathematics for economists at the introductory and advanced levels. However, there appears to be a lack of material at what we call the intermediate level, a level that takes much for granted and is concerned with proofs as well as rigorous applications, but that does not seek the abstraction and rigour of the sort of treatments that may be found in pure mathematics textbooks. Indeed, we have been unable to find a single book that covers the material herein at the level and in the way that our teaching required. Hence the present volume.

The book does not aim to be comprehensive. Rather, it contains material that we feel is particularly important to have covered before final-year work begins, and that can be covered comfortably in a standard academic year of three terms or two semesters. Other more specialized topics can be covered later, perhaps as part of a student's final year, as is possible in our programme. Thus we focus in Part I on the mathematics of linear algebra, difference equations, vector calculus and optimization. The mathematical treatment of these topics is quite detailed, but numerous worked examples are also provided. Part II of the book is devoted to a selection of applications from economics and finance. These include deterministic and stochastic dynamic macroeconomic models, input–output analysis, some probability, statistics, quadratic programming and econometric methodology, single- and multi-period choice under certainty, including general equilibrium and the term structure of interest rates, choice under uncertainty, and topics from portfolio theory, such as the capital asset pricing model.

It is assumed that most readers will have already completed good introductory courses in mathematics, economics and/or finance. The section on notation and preliminaries (pp. xix–xxiii) lists many of the basic ideas from these areas with which it is assumed that most readers will have some familiarity.

These are not rigid prerequisites, as the material presented is reasonably self-contained. However, the development is often rather rapid and the discussion fairly advanced at times. Therefore, most students, we feel, will find the book quite challenging but, we hope, correspondingly rewarding as the contents are mastered. Those who have solid preparations in mathematics, economics and finance should not be lulled into a false sense of security by the familiarity of the early material in some of the chapters; and those who have not taken such preliminary courses should be prepared for an amount of additional background reading from time to time. The exercises at the end of each chapter provide additional insight into some of the proofs, and practice in the application of the mathematical methods discussed. A solution manual will be available.

Economics may be defined as the scientific study of optimal decision-making under resource constraints and uncertainty. It is less about forecasting – however much that may be the popular perception – than it is about reacting optimally to the best available forecasts. Macroeconomics is concerned with the study of the economy as a whole. It seeks to explain such things as the determinants of the level of aggregate output (national income), the rate of growth of aggregate output, the general level of prices, and inflation, i.e. the rate of growth of prices. Microeconomics is fundamentally about the allocation of wealth or expenditure among different goods or services, which via the interaction of consumers and producers determines relative prices. Basic finance, or financial economics, is about the allocation of expenditure across two or more time periods, which gives the term structure of interest rates. A further problem in finance is the allocation of expenditure across (a finite number or a continuum of) states of nature, which yields random variables called rates of return on risky assets. Clearly, we might try to combine the concerns of microeconomics and finance to produce a rather complex problem, the solution of which points up the crucial role of mathematics.

It is probably fair to say that economics, particularly financial economics, has in recent years become as important an application of mathematics as theoretical physics. Some would say it is just another branch of applied mathematics. In mathematics departments that have traditionally taught linear algebra courses with illustrations from physics but have also followed the modern trend of offering joint programmes with economics or finance departments, the present work could be used as the basis for a general course in linear algebra for all students. The successful application of techniques from both mathematics and physics to the study of economics and finance has been outlined in the popular literature by authors such as those listed at the start of this preface. It is hoped that this book goes some way to providing students with the wherewithal to do successful mathematical work in economics and finance.

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Our thanks are also due to several cohorts of undergraduate students at Trinity College Dublin, some of whom drew attention to various typographical errors in previous versions of parts of the book used as course notes (they know who they are); and to Vahagn Galstyan for a number of useful comments on some of the linear algebra chapters.

We are grateful to three anonymous reviewers who read an earlier version of our manuscript and provided many helpful comments and suggestions; in particular to the reviewer whose detailed observations have greatly improved our treatment of Stein's lemma (Theorem 13.7.1) and Siegel's paradox (Theorem 13.10.2). We remain entirely responsible for the current version of the book, of course.

For biographical notes not otherwise attributed we have drawn on the official website of the Nobel Prize (http://nobelprize.org/nobel_prizes/economics/laureates); the MacTutor History of Mathematics archive at the University of St. Andrews (http://www-history.mcs.st-andrews.ac.uk/); the Earliest Known Uses of Some of the Words of Mathematics website (http://jeff560.tripod.com/mathword.html) and the Mathematics Genealogy Project (http://genealogy.math.ndsu.nodak.edu/).

We thank those at Routledge who have helped to bring this project to fruition, especially Simon Holt, Robert Langham, Lisa Salonen, Emily Senior and Thomas Sutton. We also thank Geoff Amor, our copyeditor, Gareth Toye, our cover designer, and Jessica Stock and her team at Sunrise Setting Ltd.

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We are especially grateful to Professor Harry M. Markowitz, Nobel laureate, for kindly contributing the Foreword.

Last, but not least, thanks to June Harrison for all the dinners.

List of abbreviations

AR autoregressive
BLUE best linear unbiased estimator
CAPM capital asset pricing model
CARA constant absolute risk aversion
cdf cumulative distribution function
CES constant elasticity of substitution
CRRA constant relative risk aversion
DARA decreasing absolute risk aversion
DCF discounted cash flow
DRRA decreasing relative risk aversion
EMH efficient markets hypothesis
EU European Union
EUR euro
GBP pound sterling
GLS generalized least squares
HARA hyperbolic absolute risk aversion
$\boldsymbol{I(0)}$ integrated to order zero
IARA increasing absolute risk aversion
iff if and only if
iid independent and identically distributed
IRR internal rate of return
IRRA increasing relative risk aversion
IS investment and saving equilibrium
ISO International Organization for Standardization
LM liquidity preference and money supply equilibrium
m million
MVN multivariate normal
NPV net present value
OLS ordinary least squares
pdf probability density function
RLS restricted least squares
rv random variable *or* random vector
s.t. so that *or* subject to *or* such that
UK United Kingdom
US(A) United States (of America)
VAR vector autoregressive
VNM von Neumann–Morgenstern
WLS weighted least squares

Notation and preliminaries

We assume familiarity with basic mathematics from Pythagoras of Samos (*c*.569–*c*.475BC) and his famous theorem onwards. Basic knowledge of probability and statistics, as well as micro- and macroeconomics and especially financial economics, would also be advantageous. The material on probability and statistics in Chapter 13 aims to be completely self-contained, but will represent quite a steep learning curve for readers who do not have some preliminary training in these areas. Similarly, some prior exposure to economics and finance will enable readers to proceed more rapidly through the other applications chapters.

For the sake of clarity, we gather together in these preliminary pages some of the more important mathematical and statistical concepts with which we assume prior familiarity, concentrating especially on those for which notational conventions unfortunately vary from one textbook to another.

Mathematical and technical terms appear in boldface where they are first introduced or defined; the corresponding page number in the index is also in boldface.

Economics students who are new to formal mathematics should be aware of common pitfalls of flawed logic, in particular with the importance of presenting the parts of a definition in the correct order and with the process of proving a theorem by arguing from the assumptions to the conclusions. Familiarity with various approaches to proofs is assumed, though the principles of proof by contradiction, proof by contrapositive and proof by induction are described when these methods are first used.[1] Similarly, mathematics students, who may be familiar with many of the mathematics topics covered, should think about the nature, subject matter and scientific methodology of economics before starting to work through the book.

Readers should be familiar with the expressions "such that" and "subject to" (both often abbreviated "s.t.") and "if and only if" (abbreviated as "iff"), and also with their meanings and use. The symbol $\forall$ is mathematical shorthand for "for all" and $\exists$ is mathematical shorthand for "there exists". The expression iff signifies a necessary and sufficient condition, or equivalence. Briefly, Q is **necessary** for P if P implies Q; and similarly P is **sufficient** for Q if P implies Q. Furthermore, P implies Q if and only if the **contrapositive**, "not Q" implies "not P", is true. We shall sometimes use an alternative symbol for a necessary and sufficient condition, namely $\Leftrightarrow$, which signifies that the truth of the left-hand side implies the truth of the right-hand side and vice versa, and also that the falsity of the left-hand side implies the falsity of the right-hand side and vice versa. Other logical symbols used are $\Rightarrow$, which means "implies", and $\Leftarrow$, which means "is implied by" or "follows from".

We will make frequent use of the identity symbol, $\equiv$, particularly in definitions; of $\approx$, which denotes "approximately equal"; and of $\sum_{i=1}^{n} x_i$ and $\prod_{j=1}^{n} x_j$ to denote the sum of, and the product of, n numbers $x_1, x_2, \ldots, x_n$, respectively; i.e.

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

and

$$\prod_{j=1}^{n} x_j = x_1 \times x_2 \times \cdots \times x_n$$

When the upper and lower limits are clear from the context, we occasionally just write $\sum_i$ or $\prod_j$.

The sum of the first n terms of a **geometric series** or **geometric progression** with first term a and **common ratio** ϕ is

$$\sum_{i=1}^{n} a\phi^{i-1} = a + a\phi + a\phi^2 + \cdots + a\phi^{n-1} = \begin{cases} \dfrac{a(1-\phi^n)}{1-\phi} = \dfrac{a(\phi^n - 1)}{\phi - 1} & \text{if } \phi \neq 1 \\ n \times a & \text{if } \phi = 1 \end{cases}$$

If $-1 < \phi < 1$, then the sum to infinity of the series is $a/(1-\phi)$. If $\phi \leq -1$ (and $a \neq 0$), then the sum oscillates without converging as $n \to \infty$. If $\phi \geq 1$, then the sum goes to $\pm\infty$ as $n \to \infty$, depending on the sign of a.

The expression $n!$, referred to as n **factorial**, denotes the product of the integers from 1 to n, inclusive, and 0! is defined to be unity; i.e. $n! \equiv \prod_{i=1}^{n} i = 1 \times 2 \times \cdots \times n$ and $0! \equiv 1$.

Readers are assumed to be familiar with basic set notation and Venn diagrams.[2] If X is the **universal set** and $B \subseteq X$, i.e. B is a **subset** of X, then $X \setminus B$ denotes the **complement** of B or $X \setminus B \equiv \{x \in X : x \notin B\}$. We say that sets B and C are **disjoint** if $B \cap C = \{\ \}$, i.e. if the intersection of B and C is the **null set** or **empty set**. The **Cartesian product** of the n sets $X_1, X_2, \ldots, X_n$ is the set of **ordered *n*-tuples**, $(x_1, x_2, \ldots, x_n)$, where the ith component, x_i, of each n-tuple is an element of the ith set, X_i.

We assume knowledge of the sets and use of natural numbers, $\mathbb{N}$, integers, $\mathbb{Z}$, and real and complex numbers, $\mathbb{R}$ and $\mathbb{C}$. Throughout the book, italic letters such as x denote specific numbers in $\mathbb{R}$. The Cartesian product $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$, denoted by $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{R}\}$, is called **(Euclidean) *n*-space**. Points in $\mathbb{R}^n$ (and sometimes in an arbitrary vector or metric space X) are denoted by lower-case boldface letters, such as $\mathbf{x}$, while an upper-case boldface letter, such as $\mathbf{X}$, will generally denote a matrix. Any $\mathbf{x} \in \mathbb{R}^n$ can also be written as the n-tuple $(x_1, x_2, \ldots, x_n)$, where $x_1, x_2, \ldots, x_n$ are referred to as the **(Cartesian) coordinates**[3] of $\mathbf{x}$. A tilde over a symbol will be used to denote a random variable (e.g. $\tilde{x}$) or a random vector (e.g. $\tilde{\mathbf{x}}$). The notation $\mathbb{R}^n_+ \equiv \{\mathbf{x} \in \mathbb{R}^n : x_i \geq 0,\ i = 1, 2, \ldots, n\}$ denotes the **non-negative orthant** of $\mathbb{R}^n$, and $\mathbb{R}^n_{++} \equiv \{\mathbf{x} \in \mathbb{R}^n : x_i > 0,\ i = 1, \ldots, n\}$ denotes the **positive orthant**.

The interval $[a, b] \equiv \{x \in \mathbb{R} : a \leq x \leq b\}$ is called a **closed interval** and $(a, b) \equiv \{x \in \mathbb{R} : a < x < b\}$ is called an **open interval**. The context will generally allow readers to distinguish between the 2-tuple $(a, b) \in \mathbb{R}^2$ and the open interval $(a, b) \subset \mathbb{R}$.

The most important result on complex numbers relied on is de Moivre's theorem,[4] which allows us to write $(\cos\theta + i\sin\theta)^t$ as $\cos t\theta + i\sin t\theta$ and $(\cos\theta - i\sin\theta)^t$ as $\cos t\theta - i\sin t\theta$, where $i \equiv \sqrt{-1}$.

Recall also that the **conjugate** of the complex number $z = a + ib$ is $\bar{z} = a - ib$, and that the conjugate of a sum is the sum of the conjugates and the conjugate of a product is the product of the conjugates. Also, the **modulus** of z is the positive square root $|z| = \sqrt{a^2 + b^2} = \sqrt{z\bar{z}}$.

The fundamental theorem of algebra states that a polynomial of degree n with real (or complex) coefficients has exactly n, possibly complex, roots, with the complex roots coming in conjugate pairs and allowing for the possibility of several roots having the same value.

The following definitions relating to functions and relations are important.

DEFINITION 0.0.1 A **function** (or **map** or **mapping**) $f: X \to Y : x \mapsto f(x)$ from a **domain** X to a **co-domain** Y is a rule that assigns to each element of the set X a unique element $f(x)$ of the set Y called the **image** of x.

DEFINITION 0.0.2 If $f: X \to Y$ and $g: Y \to Z$, then the **composition of functions** or **composite function** or **function of a function** $g \circ f: X \to Z$ is defined by $g \circ f(x) = g(f(x))$.

DEFINITION 0.0.3 A **correspondence** $f: X \to Y$ from a **domain** X to a **co-domain** Y is a rule that assigns to each element of X a non-empty subset of Y.

DEFINITION 0.0.4 The **range** of the function $f: X \to Y$ is the set $f(X) = \{f(x) \in Y : x \in X\}$.

DEFINITION 0.0.5 The function $f: X \to Y$ is **injective** (one-to-one) if and only if $f(x) = f(x') \Rightarrow x = x'$.

DEFINITION 0.0.6 The function $f: X \to Y$ is **surjective** (onto) if and only if $f(X) = Y$.

DEFINITION 0.0.7 The function $f: X \to Y$ is **bijective** (or **invertible**) if and only if it is both injective and surjective.

An invertible function $f: X \to Y$ has a well-defined inverse function $f^{-1}: Y \to X$ with $f(f^{-1}(y)) = y$ for all $y \in Y$ and $f^{-1}(f(x)) = x$ for all $x \in X$.

For any function $f: X \to Y$, if $A \subseteq X$, then

$$f(A) \equiv \{f(x) : x \in A\} \subseteq Y$$

and if $B \subseteq Y$, the notation f^{-1} is also used to denote

$$f^{-1}(B) \equiv \{x \in X : f(x) \in B\} \subseteq X$$

If f is invertible and $y \in Y$, then $f^{-1}(\{y\}) = \{f^{-1}(y)\}$. If f is not invertible, then $f^{-1}(\{y\})$ can be empty or have more than one element, but $f^{-1}: f(X) \to X$ still defines a correspondence.

DEFINITION 0.0.8 If $f: X \to Y$ is a differentiable function ($X, Y \subseteq \mathbb{R}$), then $f': X \to \mathbb{R}$ denotes the **derivative** of f, i.e. $f'(x)$ is the derivative of f at x, also occasionally denoted $\frac{df}{dx}(x)$ or dy/dx if it is known that $y = f(x)$.

Definition 0.0.9 The function $f: X \to Y$ is **homogeneous of degree** k ($k \in \mathbb{R}$) if and only if $f(\theta x) = \theta^k f(x)$ for all $\theta \in \mathbb{R}$.

When $k = 1$ and f is homogeneous of degree one, the function is sometimes called **linearly homogeneous**.

Definition 0.0.10 A **binary relation** R on the set X is a subset R of $X \times X$ or a collection of pairs (x, y) where $x \in X$ and $y \in X$.

If $(x, y) \in R$, we usually write xRy.[5]

Definition 0.0.11 The following properties of a binary relation R on a set X are often of interest:

(a) A relation R is **reflexive** if and only if xRx for all $x \in X$.
(b) A relation R is **symmetric** if and only if $xRy \Rightarrow yRx$.
(c) A relation R is **transitive** if and only if xRy and $yRz \Rightarrow xRz$.
(d) A relation R is **complete** if and only if, for all $x, y \in X$, either xRy or yRx (or both); in other words, a complete relation orders the whole set.
(e) An **equivalence relation** is a relation that is reflexive, symmetric and transitive. An equivalence relation partitions X in a natural way into disjoint equivalence classes.

In consumer theory, we will consider the weak preference relation $\succeq$, where $\mathbf{x} \succeq \mathbf{y}$ means that either the consumption bundle $\mathbf{x}$ is preferred to $\mathbf{y}$ or the consumer is indifferent between the two, i.e. that $\mathbf{x}$ is at least as good as $\mathbf{y}$.

We expect readers to have a sound knowledge of basic calculus, including the taking of limits and single-variable differentiation and integration. We assume familiarity with the definition of a derivative in terms of a limit, and with the single-variable versions of the chain rule and the product rule. We also assume knowledge of l'Hôpital's rule,[6] which states that, if the limits of the numerator and denominator in a fraction are both zero or both infinite, then the limit of the original ratio equals the limit of the ratio of the derivative of the numerator to the derivative of the denominator.

Familiarity with integration by substitution and integration by parts and with the standard rules for differentiation and integration of scalar-valued functions, in particular polynomial and trigonometric functions, is assumed.[7]

Among the trigonometric identities used later are

- the cosine rule

$$a^2 = b^2 + c^2 - 2bc\cos A$$

- the double-angle formula

$$\cos 2A = 2\cos^2 A - 1$$

- the fundamental identity

$$\cos^2 A + \sin^2 A = 1$$

An ordered arrangement of r objects from a set of n objects is called a **permutation**, and the number of different permutations of r objects that can be chosen from a set of n objects, denoted ${}^nP_r$, is given by

$$ {}^nP_r = \frac{n!}{(n-r)!} $$

A selection of r objects from a set of n objects without regard for their order is called a **combination**, and the number of different combinations of r objects that can be chosen from a set of n objects, denoted ${}^nC_r$, is given by

$$ {}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{(n-r)!\,r!} $$

We expect students to be comfortable with the properties of the exponential function $e\colon \mathbb{R} \to \mathbb{R}_{++}\colon x \mapsto e^x$, where $e \approx 2.7182\ldots$; and with its inverse, the natural logarithm function $\ln\colon \mathbb{R}_{++} \to \mathbb{R}\colon x \mapsto \ln x$; and also with the use of logarithms to any base. In particular, we rely on the fact that

$$ \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r $$

This is sometimes used as the definition of e, but others[8] prefer to start with

$$ e^r \equiv 1 + r + \frac{r^2}{2!} + \frac{r^3}{3!} + \cdots = \sum_{j=0}^{\infty}\frac{r^j}{j!} $$

The notation $|X|$ denotes the number of elements in the set X, or the **cardinality** of X, while $|z|$ denotes the modulus of the (complex) number z and $|\mathbf{X}|$ denotes the determinant of the matrix $\mathbf{X}$, more often denoted $\det(\mathbf{X})$. The modulus is just the absolute value when z is real rather than complex. There is obviously some potential for confusion from use of the same symbol for three different concepts, but the context and notation within the symbol will almost always make the distinctions clear.

The collection of all possible subsets of the set X, or the **power set** of X, is denoted by 2^X. Note that $|2^X| = 2^{|X|}$.

The least upper bound or **supremum** of a set, X, of real numbers, denoted $\sup(X)$, is the smallest real number that is greater than or equal to every number in the set. For example, $\sup\{1, 2, 3, 4\} = 4$ and $\sup\{x \in \mathbb{R}^n\colon 0 < x < 1\} = 1$. The second of these examples indicates that the supremum is not necessarily the maximum real number in the set.

The greatest lower bound or **infimum** of a set, X, of real numbers, denoted $\inf(X)$, is the largest real number that is less than or equal to every number in the set. For example, $\inf\{1, 2, 3, 4\} = 1$, $\inf\{x \in \mathbb{R}^n\colon 0 \le x \le 1\} = 0$ and $\inf\{x \in \mathbb{R}^n\colon x^3 > 2\} = 2^{1/3}$. The infimum is not necessarily the minimum real number in a set.