# The Development of Mathematics

E.T. Bell

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# THE DEVELOPMENT OF MATHEMATICS

E. T. Bell

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# To Any Prospective Reader

Nearly fifty years ago an American critic, reviewing the first volume (1888) of Lie's *Theorie der Transformationsgruppen*, set his own pace (and ours) in the following remarks.

> There is probably no other science which presents such different appearances to one who cultivates it and one who does not, as mathematics. To [the noncultivator] it is ancient, venerable, and complete; a body of dry, irrefutable, unambiguous reasoning. To the mathematician, on the other hand, his science is yet in the purple bloom of vigorous youth, everywhere stretching out after the "attainable but unattained," and full of the excitement of nascent thoughts; its logic is beset with ambiguities, and its analytic processes, like Bunyan's road, have a quagmire on one side and a deep ditch on the other, and branch off into innumerable by-paths that end in a wilderness. [a]

Once we venture beyond the rudiments, we may agree that those who cultivate mathematics have more interesting things to say than those who merely venerate. Accordingly, we shall follow the cultivators in their explorations of a Bunyan's road through the development of mathematics. If occasionally we have no eyes for the purple bloom, it will be because we shall need all our faculties t avoid falling into the ditch or wandering off into a wilderness of trivialities that might be mistaken fo mathematics or for its history. And we shall leave to antiquarians the difficult and delicate task of restoring the roses to the cheeks of mathematical mummies.

The course chosen in the following chapters was determined by two factors. The first was the request from numerous correspondents, principally students and instructors, for a broad account of th general development of mathematics, with par-ticular reference to the main concepts and methods th have, in some measure, survived. The second was personal association for several years with creative mathematicians in both the pure and the applied divisions.

*Not* a history of the traditional kind, but a narrative of the decisive epochs in the development of mathematics was wanted. A large majority asked for technical hints, where possible without too grea detail, why certain things continue to interest mathematicians, technologists, and scientists, while others are ignored or dismissed as being no longer vital. Many who planned to end their mathematica education with the calculus, or even in some instances earlier, wished to be shown something of the general development of mathematics beyond that outstanding landmark of seventeenth-century thought, as part of a civilized education. Those intending to continue in mathematics or science or technology also asked for a broad general treatment with technical hints. They gave two additional reasons, the second of singular interest to any professed teacher. They believed that a survey of the main directions along which living mathematics has developed would enable them to decide more intelligently in what particular field of mathematics, if any, they might find a lasting satisfaction.

The second reason for their request was characteristic of a generation that has grown rather tired o being ~~told what to think and whom to respect. These candid young critics of their would-be~~ educators hoped that a cursory personal inspection of the land promised them, even from afar off, would enable them to resist the blandishments of persuasive subdivides' bent on selling their own tracts to the inexperienced. We seem to have come a long way since 1873, when that erudite English historian of mathematics and indefatigable manufacturer of drier-than-dust college textbooks, Isaac Todhunter (1820-1884), counseled a meek docility, sustained by an avid credulity, as the path of intellectual rectitude:

> If he [a student of mathematics] does not believe the statements of his tutor, probably [in Todhunter's day at Cambridge] a clergyman of mature knowledge, recognized ability and blameless character-his suspicion is irrational, and manifests a want of the power of appreciating evidence, a want fatal to his success in that branch of science which he is supposed to be cultivating.

Be the wisdom of Todhunter's admonition what it may, it is astonishing how few students entering serious work in mathematics or its applications have even the vaguest idea of the highways, the pitfalls, and the blind alleys ahead of them. Consequently, it is the easiest thing in the world for an enthusiastic teacher, "of mature knowledge, recognized ability and blameless character," to sell his misguided pupils a subject that has been dead for forty or a hundred years, under the sincere delusion that he is disciplining their minds. With only the briefest glimpse of what mathematics in this twentieth century —not in 2100 B.C.—is about, any student of normal intelligence should be able to distinguish between live teaching and dead mathematics. He will then be less likely than his confidin companion to drown in the ditch or perish in the wilderness.

Many asked for some reference to the social implications of mathematics. A classic strategy in mathematics is the reduction of an unsolved problem to one already solved. It seems plausible that more than half the problem of mathematics and society is reducible to that of the physical sciences and society. There being as yet no widely accepted solution of the latter problem, we shall leave the former with the reduction indicated. Anyone will thus be able to reach his own conclusions from that solution of the scientific problem which he accepts. Proposed solutions range from Platonic realism a one extreme to Marxian determinism at the other. Occasional remarks may suggest an inquiry into th equally difficult question of what part civilization, with its neuroses, its wars, and its national jealousies, has played in mathematics. These asides may be of interest to those intending to make mathematics their lifework. Incidentally, in this connection, I was told that I might write for adults. Chronological age is not necessarily a measure of adulthood; a first-year student in a university may be less infantile, in everything but mathematics, than the distinguished savant lecturing at him.

The topics selected for description were chosen after consultation with numerous professionals wh know from hard personal experience what mathematical invention means. On their advice, *only main trends* of the past six thousand years are considered, and these are presented *only through typical major episodes* in each. As might be anticipated by any worker in mathematics, the conclusions reached by following such advice differ occasionally from those hallowed by the purely historical tradition. Wherever this is so, references to other accounts will enable any reader to form his own opinion. There are no absolutes (except possibly this) in mathematics or in its history.

Most of the differences reflect two possible and sometimes divergent readings of mathematical evolution. Whoever has himself attempted to advance mathematics is inclined to be more skeptical than the average spectator toward any alleged anticipation of notable progress. From his own experience and that of others still living, the professional mathematician suspects that often what looks like an anticipation after the advance was made was not even aimed in the right direction. Fron many a current instance, he knows further that when at length progress started, it proceeded along lines totally different from those which, in retrospect, it 'should' have followed.

Nothing is easier, on the other hand, than to fit a deceptively smooth curve to the discontinuities o mathematical invention. Everything then appears as an orderly progression from the Egypt of 4000 B.C. and the Babylon of 2000 B.C. to the Göttingen of 1934 and the U.S.A. of 1945, with Cavalieri, fo instance, indistinguishable from Newton in the neighborhood of the calculus, or Lagrange from Fourier in that of trigonometric series, or Bhaskara from Lagrange in the region of Fermat's equation Professional historians may sometimes be inclined to overemphasize the smoothness of the curve; professional mathematicians, mindful of the dominant part played in geometry by the singularities o curves, attend to the discontinuities. This is the origin of most differences of opinion between the majority of those who cultivate mathematics and the majority of those who do not. That such differences should exist is no disaster. Dissent is good for the souls of all concerned.

No apology need be tendered the thousands of dead and living mathematicians whose names are no mentioned. Only a meaningless catalogue could have cited a tenth of those who have created mathematics. Nor, when between 4,000 and 5,000 papers and books devoted to mathematical researc —the creation of new mathematics—are being published every 365 days, is there any point in attempting to minimize the omission of certain topics that have interested, and may still interest, hundreds of these unnamed thousands. However, what a sufficient number of competent men conside the vital things are at least mentioned. Anyone desirous of following the detailed history of certain major developments will find the technical histories of special topics, written by mathematicians for mathematicians, ample for a beginning. Some of these severely technical histories extend to hundred of pages, a few to thousands ; they refer to the labors of thousands of men, most of whom are all but completely forgotten. Yet, like the tiny creatures whose empty frames survive in massive coral reefs that can wreck a battleship, these hordes of all but anonymous mathematicians have left something i the structure of mathematics more durable than their own brief and commonplace lives.

As to the mechanical features of the book, the inevitable footnotes have been kept to a minimum b the simple expedient of throwing hundreds away. Some direct those seeking further information on t relevant mathematics to works by creative mathematicians. Other things being equal, preference is given works containing extensive bibliographies compiled by experts having firsthand knowledge of the subjects treated. *A superscript number indicates a footnote; all are collected for easy reference just before the index. All should be ignored till a possible return to some point.*

The index will be found helpful. Men's initials and dates (except a very few, unobtainable without undue labor), seldom repeated in the text, are given in the index; cross references to definitions, etc., are avoided by the same means.

Nationalities are stated; if more than one country has a claim to some man, the place where he did most of his work is given. On a previous occasion *(Men of mathematics,* Simon & Schuster, New York, 1937), I almost precipitated an international incident by calling a Pole a Russian. I trust that fe such disastrous blunders will be found here. The book mentioned contains full-length biographies of about thirty-five leading mathematicians of the past.

Dates in the text appended to mathematical events serve two purposes, the first of which is obviou

The second is to avoid elaborate references. The date, if later than 1636 and earlier than 1868, will usu~~ally enable anyone seriously interested to locate the matter concerned in the collected works~~ of th author cited; if later than 1867, and whether or not collected works are available, the exact reference, with a concise abstract of the work, is given in the annual *Jahrbuch über die Fortschritte der Mathematik*. For the period beginning In 1931, the *Zentralblatt für Mathematik und ihre Grenzgebiet* serves the same purpose. The American *Mathematical Reviews*, 1940-, is of the same general character as the German abstract journals. Comparatively scarce early periodicals, likely to be found only in specialized libraries, are not cited, although they were frequently consulted. This omission may be partly compensated by referring to the German and French mathematical encyclopedias liste in the notes. Other references to sources before 1637 are given in the proper places.

For the period before 1637, the works of professional historians of mathematics have been used fo some matters on which the historians are in approximate agreement among themselves. Theirs is a difficult and exacting pursuit; and if controversies over the trivia of mathematics, of but slight intere to either students or professionals, absorb a considerable part of their energies, the residue of apparently sound facts no doubt justifies the inordinate expense of obtaining it. Without the devoted labors of these scholars, mathematicians would know next to nothing, and perhaps care less, about th first faltering steps of their science. Indeed, an eminent French analyst of the twentieth century declared that neither he nor any but one or two of his fellow professionals had the slightest interest i the history of mathematics as conceived by historians. He amplified his statement by observing that the only history of mathematics that means anything to a mathematician is the thousands of technica papers cramming the journals devoted exclusively to mathematical research. These, he averred, are the true history of mathematics, and the only one either possible or profitable to write. Fortunately, I am not attempting to write a history of mathematics; I hope only to encourage some to go on, and decide for themselves whether the French analyst was right.

Preference has been given in citing purely historical references to works in English, French, or German, as these are the three languages of which those interested in mathematics are most likely to have an adequate reading knowledge. For those especially interested in geometry, Italian also is necessary. Italian historical works are included in the bibliographical material of the histories listed.

To the many professional friends who have advised me on their respective specialties and whose generous help I have attempted to pass on to others, I am very grateful. A special word of thanks is due Professor W. H. Gage, of the University of British Columbia, who removed many obscurities and greatly improved several of the presentations.

This has been an opportunity to do something a little off the beaten track to show prospective readers how the mathematics familiar to them got where it is, and where it is going from there. I trus that students will tolerate the departure from the traditional textbook. For one thing, at any rate, the more sensible should be grateful: only the most ingenious instructor could set an examination on the book.

It has, unhappily, been necessary in writing the book to consider many things besides the masterpieces of mathematics. Rising from a protracted and not always pleasant session with the work of bickering historians, scholarly pedants, and contentious mathematicians, often savagely contradicting or meanly disparaging one another, I pass on, for what it may be worth, the principal thing I have learned to appreciate as never before. It is contained in Buddha’s last injunction to his followers:

*Believe nothing on hearsay. Do not believe in traditions because they are old, or in anything on th* *mere authority of myself or any other teacher*.

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E. T. BEL

## NOTE TO THE SECOND EDITION

About fifty pages of new material have been added in this edition. The additions include numerous short amplifications of miscellaneous topics from Greek mathematics to mathematical logic, with longer notes on symbolism, algebraic and differential geometry, lattices, and other subjects in which there have been striking recent advances.

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CALIFORNIA INSTITUTE OF TECHNOLOGY,
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# Table of Contents

# CHAPTER 1

## *General Prospectus*

In all historic times all civilized peoples have striven toward mathematics. The prehistoric origins are as irrecoverable as those of language and art, and even the civilized beginnings can only be conjectured from the behavior of primitive peoples today. Whatever its source, mathematics has com down to the present by the two main streams of number and form. The first carried along arithmetic and algebra, the second, geometry. In the seventeenth century these two united, forming the ever-broadening river of mathematical analysis. We shall look back in the following chapters on this grea river of intellectual progress and, in the diminishing perspective of time, endeavor to see the more outstanding of those elements in the general advance from the past to the present which have endured

'Form,' it may be noted here to prevent a possible misapprehension at the outset, has long been understood mathematically in a sense more general than that associated with the shapes of plane figures and solid bodies. The older, geometrical meaning is still pertinent. The newer refers to the structure of mathematical relations and theories. It developed, not from a study of spacial form as such, but from an analysis of the proofs occurring in geometry, algebra, and other divisions of mathematics.

Awareness of number and spacial form is not an exclusively human privilege. Several of the highe animals exhibit a rudimentary sense of number, while others approach genius in their mastery of for Thus a certain cat made no objection when she was relieved of two of her six kittens, but was plainly distressed when she was deprived of three. She was relatively as advanced arithmetically as the savages of an Amazon tribe who can count up to two, but who confuse all greater numbers in a nebulous 'manly.'

Again, the intellectual rats that find their way through the mazes devised by psychologists are passing difficult examinations in topology. At the human level, a classic puzzle which usually suffic to show the highly intelligent the limitations of their spacial intuition is that of constructing a surfac with only one side and one boundary.

Although human beings and the other animals thus meet on a common ground of mathematical sense, mathematics as it has been understood for at least twenty-five centuries is on a far higher plan of intelligence.

### *Necessity for proof; emergence of mathematics*

Between the workable empiricism of the early land measurers who parceled out the fields of ancie Egypt and the geometry of the Greeks in the sixth century before Christ there is a great chasm. On th remoter side lies what preceded mathematics, on the nearer, mathematics; and the chasm is bridged b deductive reasoning applied consciously and deliberately to the practical inductions of daily life. Without the strictest deductive proof from admitted assumptions, explicitly stated as such, mathematics does not exist. This does not deny that intuition, experiment, induction, and plain ~~guessing are important elements in mathematical invention. It merely states the criterion by which~~ th final product of all the guessing, by whatever name it be dignified, is judged to be or not to be mathematics. Thus, for example, the useful rule, known to the ancient Babylonians, that the area of a rectangular field can be computed by 'length times breadth,' may agree with experience to the utmos refinement of physical measurement; but the rule is not a part of mathematics until it has been deduced from explicit assumptions.

It may be significant to record that this sharp distinction between mathematics and other sciences began to blur slightly under the sudden impact of a greatly accelerated applied mathematics, so calle in the second world war. Semiempirical procedures of calculation, certified by their pragmatic utility in war, were accorded full mathematical prestige. This relaxation of traditional demands brought the resulting techniques closer in both method and spirit to engineering and the physical sciences. It was acclaimed by some of its practitioners as a long-overdue democratization of the most aristocratic of the sciences. Others, of a more conservative persuasion, deplored the passing of the ideal of strict deduction, as a profitless confusion of a simple issue which had at last been clarified after several centuries of futile disputation. One fact, however, emerged from the difference of opinion: It is difficult, in modern warfare, to wreck, to maim, or to kill efficiently without a considerable expenditure of mathematics, much of which was designed originally for the development of those sciences and arts which create and conserve rather than destroy and waste.

It is not known where or when the distinction between inductive inference-the summation of raw experience—and deductive proof from a set of postulates was first made, but it was sharply recognized by the Greek mathematicians as early as 550 B.C. As will appear later, there may be som grounds for believing that the Egyptians and the Babylonians of about 2000 B.C. had recognized the necessity for deductive proof. For proof in even the rough and unready calculations of daily life is indeed a necessity, as may be seen from the mensuration of rectangles.

If a rectangle is 2 feet broad and 3 long, an easy proof sustains the verdict of experience, founded o direct measurement, that the area is 6 square feet. But if the breadth is $\sqrt{2}$ and the length $\sqrt{3}$ feet, th area cannot be determined as before by cutting the rectangle into unit squares; and it is a profoundly difficult problem to prove that the area is $\sqrt{6}$ feet, or even to give intelligible, usable meanings to $\sqrt{2}$, $\sqrt{3}$, $\sqrt{6}$, and 'area.' By taking smaller and smaller squares as unit areas, closer and closer approximations to the area are obtained, but a barrier is soon reached beyond which direct measurement cannot proceed. This raises a question of cardinal importance for a just understanding o the development of all mathematics, both pure and applied.

Continuing with the $\sqrt{2} \times \sqrt{3}$ rectangle, we shall suppose that refined measurement has given 2.4494897 as the area. This is correct to the seventh decimal, but it is not right, because $\sqrt{6}$, the exac area, is not expressible as a terminated decimal fraction. If seven-place accuracy is the utmost demanded, the area has been found. This degree of precision suffices for many practical applications including precise surveying. But it is inadequate for others, such as some in the physical sciences and modern statistics. And before the seven-place approximation can be used intelligently, its order of error must be ascertained. Direct measurement cannot enlighten us; for after a certain limit, quickly passed, all measurements blur in a common uncertainty. Some universal agreement on what is meant by the exact area must be reached before progress is possible. Experience, both practical and theoretical, has shown that a consistent and useful mensuration of rectangles is obtained when the rul ‘length times breadth’ is deduced from postulates abstracted from a lower level of experience and accepted as valid. The last is the methodology of all mathematics.

Mathematicians insist on deductive proof for practically workable rules obtained inductively because they know that analogies between phenomena at different levels of experience are not to be accepted at their face value. Deductive reasoning is the only means yet devised for isolating and examining hidden assumptions, and for following the subtle implications of hypotheses which may b less factual than they seem. In its modern technical uses of the deductive method, mathematics employs much sharper tools than those of the traditional logic inherited from ancient and medieval times.

Proof is insisted upon for another eminently practical reason. The difficult technology of today is likely to become the easy routine of tomorrow; and a vague guess about the order of magnitude of an unavoidable error in measurement is worthless in the technological precision demanded by modern civilization. Working technologists cannot be skilled mathematicians. But unless the rules these men apply in their technologies have been certified mathematically and scientifically by competent experts, they are too dangerous for use.

There is still another important social reason for insistence on mathematical demonstration, as ma be seen again from the early history of surveying. In ancient Egypt, the primitive theory of land measurement, without which the practice would have been more crudely wasteful than it actually wa sufficed for the economy of the time. Crude both practically and theoretically though this surveying was, it taxed the intelligence of the Egyptian mathematicians. Today the routine of precise surveying can be mastered by a boy of seventeen; and those applications of the trigonometry that evolved from primitive surveying and astronomy which are of greatest significance in our own civilization have nc connection with surveying. Some concern mechanics and electrical technology, others, the most advanced parts of the physical sciences from which the industries of twenty or a hundred years hence may evolve.

Now, contrary to what might be supposed, modern trigonometry did not develop in response to any practical need. Modern trigonometry is impossible without the calculus and the mathematics of $\sqrt{-}$ To cite but one of the commoner applications, over a century and a half elapsed before this trigonometry became indispensable in the theory and practice of alternating currents. Long before anyone had dreamed of an electric dynamo, the necessary mathematics of dynamo design was available. It had developed largely because the analysts of the eighteenth century sought to understan mathematically the somewhat meager legacy of trigonometry bequeathed them by the astronomers o ancient Greece, the Hindus, and the mathematicians of Islam. Neither astronomy nor any other scienc of the eighteenth century suggested the introduction of $\sqrt{-1}$, which completed trigonometry, as no such science ever made any use of the finished product.

The importance of mathematics, from Babylon and Egypt to the present, as the primary source of workable approximations to the complexites of daily life is generally appreciated. In fact, a mathematician might believe it is almost too generally appreciated. It has been preached at the publi in school and out, by socially conscious educators until almost anyone may be pardoned for believing that the rule of life is rule of thumb. Because routine surveying, say, requires only mediocre intelligence, and because surveying is a minor department of applied mathematics, therefore only tha mathematics which can be manipulated by rather ordinary people is of any social value. But no growing economy can be sustained by rule of thumb. If new applications of a furiously expanding science are to be possible, difficult and abstruse mathematical theories far beyond the college level must continue to be developed by those having the requisite talents. In this living mathematics it is ima~~gination and rigorous proof which count, not the numerical accuracy of the machine shop or~~ the computing laboratory.

A familiar example from common things will show the necessity for mathematics as distinguished from calculation. A nautical almanac is one of the indispensables of modern navigation and hence of commerce. Machines are now commonly used for the heavy labor of computing. Ultimately the computations depend upon the motions of the planets, and these are calculated from the infinite (non terminating) series of numbers given by the Newtonian theory of gravitation. For the actual work of computation a machine is superior to any human brain; but no machine yet invented has had brains enough to reject nonsense fed into it. From a grotesquely absurd set of data the best of machines will return a final computation that looks as reasonable as any other. Unless the series used in dynamical astronomy converge to definite limiting numbers (asymptotic series also are used, but not properly divergent), it is futile to calculate by means of them. A table computed by properly divergent series would be indistinguishable to the untrained eye from any other; but the aviator trusting it for a flight from Boston to New York might arrive at the North Pole. Despite its inerrant accuracy and attractive appearance, even the most highly polished mechanism is no substitute for brains. The research mathematician and the scientific engineer supply the brains; the machine does the rest.

Nobody with a grain of common sense would demand a strict proof for every tentative application of complicated mathematics to new situations. Occasionally in problems of excessive difficulty, like some of those in nuclear physics, calculations are performed blindly without reference to mathematical validity; but even the boldest calculator trusts that his temerity will some day be certified rationally. This is a task for the mathematicians, not for the scientists. And if science is to b more than a midden of uncorrelated facts, the task must be carried through.

### *Necessity for abstractness*

With the recognition that strict deductive reasoning has both practical and aesthetic values, mathematics began to emerge some six centuries before the Christian era. The emergence was complete when human beings realized that common experience is too complex for accurate description.

Again it is not known when or where this conclusion was first reached, but the Greek geometers of the fourth century B.C. at latest had accepted it, as is shown by their work. Thus Euclid in that centur stated the familiar definition: “A circle is a plane figure contained by one line, called the circumference, and is such that all straight lines drawn from a certain point, called the center, within the figure to the circumference are equal.”

There is no record of any such figure as Euclid’s circle ever having been observed by any human being. Yet Euclid’s ideal circle is not only that of school geometry, but is also the circle of the handbooks used by engineers in calculating the performance of machines. Euclid’s mathematical circle is the outcome of a deliberate simplification and abstraction of observed disks, like the full moon’s, which appear ‘circular’ to unaided vision.

This abstracting of common experience is one of the principal sources of the utility of mathematic and the secret of its scientific power. The world that impinges on the senses of all but introverted solipsists is too intricate for any exact description yet imagined by human beings. By abstracting and simplifying the evidence of the senses, mathematics brings the worlds of science and daily life into focus with our myopic comprehension, and makes possible a rational description of our experiences which accords remarkably well with observation.

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Abstractness, sometimes hurled as a reproach at mathematics, is its chief glory and its surest title t practical usefulness. It is also the source of such beauty as may spring from mathematics.

## *History and proof*

In any account of the development of mathematics there is a peculiar difficulty, exemplified in the two following assertions, about many statements concerning proof.

(*A*) It is proved in Proposition 47, Book 1, of Euclid's *Elements*, that the square on the longest side of a right-angled triangle is equal to the sum of the squares on the other two sides (the so-called Pythagorean theorem).

(B) Euclid proved the Pythagorean theorem in Proposition 47 of Book I of his *Elements.*

In ordinary discourse, (*A*), (*B*) would usually be considered equivalent—both true or both false. Here *(A)* is false and (*B*) true. For a clear understanding of the development of mathematics it is important to see that this distinction is not a quibble. It is also essential to recognize that comprehension here is more important than knowing the date (c. 330-320 B.C.) at which the *Element* were written, or any other detail of equal antiquarian interest. In short, the crux of the matter is mathematics, which is at least as important as history, even in histories of mathematics.

The statement *(A)* is false because the attempted proof in the *Elements* is invalid. The attempt is vitiated by tacit assumptions that Euclid ignored in laying down the postulates from which he undertook to deduce the theorems in his geometry. From those same postulates it is easy to deduce, b irrefragable logic, spectacularly paradoxical consequences, such as "all triangles are equilateral." Thus when an eminent scholar of Greek mathematics asserts that owing to the "inerring logic" of the Greeks, "there has been no need to reconstruct, still less to reject as unsound, any essential part of their doctrine," mathematicians must qualify assent by referring to the evidence. The "essential part their doctrine" has indeed come down to us unchanged, that part being insisteiice on deductive proof But in the specific instance of Euclid's proofs, many have been demolished in detail, and it would be easy to destroy more were it worth the trouble.

The statement *(B)* is true because the validity of a proof is a function of time. The standard of mathematical proof has risen steadily since 1821, and finality is no longer sought or desired. In Euclid's day, and for centuries thereafter, the attempted proof of the Pythagorean proposition satisfie all the current requirements of logical and mathematical rigor. A sound proof today does not differ greatly in outward appearance from Euclid's; but if we inspect the postulates required to validate the proof, we notice several which Euclid overlooked. A carefully taught child of fourteen today can easily detect fatal omissions in many of the demonstrations in elementary geometry accepted as soun less than fifty years ago.

It is clear that we must have some convention regarding 'proof.' Otherwise, few historical statements about mathematics will have any meaning. Whenever in the sequel it is stated that a certa result was proved, this is to be understood for the sense as in *(B),* namely, that the proof was accepte as valid by professional mathematicians at the time it was given. If, for example, it is asserted that a work of Newton or of Euler contains a proof of the binomial theorem for exponents other than positi integers, the assertion is false for the (*A*) meaning, true for the *(B).* The proofs which these great mathematicians gave in the seventeenth and eighteenth centuries were valid *at that time,* although th would not be accepted today by a competent teacher from a student in the first college course.

It need scarcely be remarked that few modest mathematicians today expect all of their own proofs to ~~survive the criticisms of their successors unscathed. Mathematics thrives on intelligent criticism~~, and it is no disparagement of the great work of the past to point out that its very defects have inspired work as great.

Failure to observe that mathematical validity depends upon its epoch may generate scholarly but vacuous disputes over historical minutiae. Thus a meticulous historian who asserts that the Greeks of Euclid’s time failed to solve quadratic equations by their geometric method because they ‘overlooke possible negative roots, to say nothing of imaginaries, himself overlooks one of the most interesting phenomena in the entire history of mathematics.

Until positive rational fractions and negative numbers were invented by mathematicians (or ‘discovered,’ if the inventors happened to be Platonic realists), a quadratic equation with rational integer coefficients had precisely one root, or precisely two, or precisely none. A Babylonian of a sufficiently remote century who gave 4 as *the* root of $x^2 = x + 12$ had solved his equation completely, because -3, which we now say is the other root, did not exist for him. Negative numbers were not in his number system. The successive enlargements of the number system necessary to provide all algebraic equations with roots equal in number to the respective degrees of the equations was one of the outstanding landmarks in mathematical progress, and it took about four thousand years of civiliz mathematics to establish it. The final necessary extension was delayed till the nineteenth century.

An educated algebraist today, wishing to surpass the meticulous critic in pedantry, would point ou that “how many roots has $x^2 = x$?” is a meaningless question until the domain in which the roots may lie has been specified. If the domain is that of complex numbers, this equation has precisely two root 0, 1. But if the domain is that of Boolean algebra, this same quadratic (since 1847) has had *n* roots where *n* is any integer equal to, or greater than, 2. Boolean algebra, it may be remarked, is as legitimately a province of algebra today as is the theory of quadratic equations in elementary schoolbooks. In short, criticizing our predecessors because they completely solved their problems within the limitations which they themselves imposed is as pointless as deploring our own inability t imagine the mathematics of seven thousand years hence.

Some of the most significant episodes in the entire history of mathematics will be missed unless this dependence of validity upon time is kept in mind as we proceed. In ancient Greece, for example, the entire development of by far the greater part of such Greek mathematics as is still of vital interes stems from this fact. The discontinuities in the time curve of acceptable proof, where standards of rigor changed abruptly, are perhaps the points of greatest interest in the development of mathematics The four most abrupt appear to have been in Greece in the fifth century B.C., in Europe in the 1820’s and in the 1870’s, and again in Europe in the twentieth century.

None of this implies that mathematics is a shifting quicksand. Mathematics is as stable and as firmly grounded as anything in human experience, and far more so than most things. Euclid’s Proposition I, 47 stands, as it has stood for over 2,200 years. Under the proper assumptions it has bee rigorously proved. Our successors may detect flaws in our reasoning and create new mathematics in their efforts to construct a proof satisfying to themselves. But unless the whole process of mathematical development suffers a violent mutation, there will remain some proposition recognizably like that which Euclid proved in his generation.

Not all of the mathematics of the past has survived, even in suitably modernized form. Much has been discarded as trivial, inadequate, or cumbersome, and some has been buried as definitely fallacious. There could be no falser picture of mathematics than that of “the science which has never had to retrace a step.” If that were true, mathematics would be the one perfect achievement of a race ~~admittedly incapable of perfection. Instead of this absurdity, we shall endeavor to portray~~ mathematics as the constantly growing, human thing that it is, advancing in spite of its errors and partly because of them.

## *Five streams*

The picture will be clearer if its main outlines are first roughly blocked in and retained while detai are being inspected.

Into the two main streams of number and form flowed many tributaries. At first mere trickles, som quickly swelled to the dignity of independent rivers. Two in particular influenced the whole course o mathematics from almost the earliest recorded history to the twentieth century. Counting by the natural numbers 1, 2, 3, ... introduced mathematicians to the concept of *discreteness*. The invention o irrational numbers, such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{6}$; attempts to compute plane areas bounded by curves or by incommensurable straight lines; the like for surfaces and volumes; also a long struggle to give a coherent account of motion, growth, and other sensually continuous change, forced mathematicians t invent the concept of *continuity*.

The whole of mathematical history may be interpreted as a battle for supremacy between these two concepts. This conflict may be but an echo of the older strife so prominent in early Greek philosophy the struggle of the One to subdue the Many. But the image of a battle is not wholly appropriate, in mathematics at least, as the continuous and the discrete have frequently helped one another to progress.

One type of mathematical mind prefers the problems associated with continuity. Geometers, analysts, and appliers of mathematics to science and technology are of this type. The complementary type, preferring discreteness, takes naturally to the theory of numbers in all its ramifications, to algebra, and to mathematical logic. No sharp line divides the two, and the master mathematicians hav worked with equal ease in both the continuous and the discrete.

In addition to number, form, discreteness, and continuity, a fifth stream has been of capital importance in mathematical history, especially since the seventeenth century. As the sciences, beginning with astronomy and engineering in ancient times and ending with biology, psychology, and sociology in our own, became more and more exact, they made constantly increasing demands on mathematical inventiveness, and were mainly responsible for a large part of the enormous expansion of all mathematics since 1637. Again, as industry and invention became increasingly scientific after the industrial revolution of the late eighteenth and early nineteenth centuries, they too stimulated mathematical creation, often posing problems beyond the existing resources of mathematics. A current instance is the problem of turbulent flow, of the first importance in aerodynamics. Here, as in many similar situations, attempts to solve an essentially new technological problem have led to further expansions of pure mathematics.

## *The time-scale*

It will be well to have some idea of the distribution of mathematics in time before looking at individual advances.

The time curve of mathematical productivity is roughly similar to the exponential curve of biologi growth, starting to rise almost imperceptibly in the remote past and shooting up with ever greater rapidity as the present is approached. The curve is by no means smooth; for, like art, mathematics ha had ~~its depressions. There was a deep one in the Middle Ages, owing to the mathematical barbarism~~ o Europe being only partly balanced by the Moslem civilization, itself (mathematically) a sharp recession from the great epoch (third century B.C.) of Archimedes. But in spite of depressions, the general trend from the past to the present has been in the upward direction of a steady increase of valid mathematics.

We should not expect the curve for mathematics to follow those of other civilized activities, say ar and music, too closely. Masterpieces of sculpture once shattered are difficult to restore or even to remember. The greater ideas of mathematics survive and are carried along in the continual flow, permanent additions immune to the accidents of fashion. Being expressed in the one universally intelligible language as yet devised by human beings, the creations of mathematics are independent o national taste, as those of literature are not. Who today except a few scholars is interested or amused by the ancient Egyptian novelette of the two thieves ? And how many can understand hieroglyphics sufficiently to elicit from the story whatever significance it may once have had for a people dead all three thousand years ? But tell any engineer, or any schoolboy who has had some mensuration, the Egyptian rule for the volume of a truncated square pyramid, and he will recognize it instantly. Not only are the valid creations of mathematics preserved; their mere presence in the stream of progress induces new currents of mathematical thought.

The majority of working mathematicians acquainted in some measure with the mathematics create since 1800 agree that the time curve rises more sharply thereafter than before. An open mind on this question is necessary for anyone wishing to see mathematical history as the majority of mathematicians see it. Many who have no firsthand knowledge of living mathematics beyond the calculus believe on grossly inadequate evidence that mathematics experienced its golden age in some more or less remote past. Mathematicians think not. The recent era, beginning in the nineteenth century, is usually regarded as the golden age by those personally conversant with mathematics and a least some of its history.

An unorthodox but reasonable apportionment of the time-scale of mathematical development cuts all history into three periods of unequal lengths. These may be called the remote, the middle, and the recent. The remote extends from the earliest times of which we have reliable knowledge to A.D. 1637 the middle from 1638 to 1800. The recent period, that of modern mathematics as professionals today understand mathematics, extends from 1801 to the present. Some might prefer 1821 instead of 1801.

There are definite reasons for the precise dates. Geometry became analytic in 1637 with the publication of Descartes' masterpiece. About half a century later the calculus of Newton and Leibniz also the dynamics of Galileo and Newton, began to become the common property of all creative mathematicians. Leibniz certainly was competent to estimate the magnitude of this advance. He is reported to have said that, of all mathematics from the beginning of the world to the time of Newton, what Newton had done was much the better half.

The eighteenth century exploited the methods of Descartes, Newton, and Leibniz in all department of mathematics as they then existed. Perhaps the most significant feature of this century was the beginning of the abstract, completely general attack. Although adequate realization of the power of th abstract method was delayed till the twentieth century, there are notable anticipations in Lagrange's work on algebraic equations and, above all, in his analytic mechanics. In the latter, a direct, universal method unified mechanics as it then was, and has remained to this day one of the most powerful tools in the physical sciences. There was nothing like this before Lagrange.

The last date, 1801, marks the beginning of a new era of unprecedented inventiveness, opening wit the ~~publication of Gauss' masterpiece. The alternative, 1821, is the year in which Cauchy began~~ the first satisfactory treatment of the differential and integral calculus.

As one instance of the greatly accelerated productivity in the nineteenth century, consequent to a thorough mastery and amplification of the methods devised in the middle period, an episode in the development of geometry is typical. Each of five men—Lobachewsky, Bolyai, Plucker, Riemann, Lie invented as part of his lifework as much (or more) new geometry as was created by all the Greek mathematicians in the two or three centuries of their greatest activity. There are good grounds for the frequent assertion that the nineteenth century alone contributed about five times as much to mathematics as had all preceding history. This applies not only to quantity but, what is of incomparably greater importance, to power.

Granting that the mathematicians before the middle period may have encountered the difficulties attendant on all pioneering, we need not magnify their great achievements to universe-filling proportions. It must be remembered that the advances of the recent period have swept up and include nearly all the valid mathematics that preceded 1800 as very special instances of general theories and methods. Of course nobody who works in mathematics believes that our age has reached the end, as Lagrange thought his had just before the great outburst of the recent period. But this does not alter th fact that most of our predecessors did reach very definite ends, as we too no doubt shall. Their limite methods precluded further significant progress, and it is possible, let us hope probable, that a century hence our own more powerful methods will have given place to others yet more powerful.

### *Seven periods*

A more conventional division of the time-scale separates all mathematical history into seven periods:

1. From the earliest times to ancient Babylonia and Egypt, inclusive.
2. The Greek contribution, about 600 B.C. to about A.D. 300, the best being in the fourth and third centuries B.C.
3. The oriental and Semitic peoples—Hindus, Chinese, Persians, Moslems, Jews, etc., partly before partly after (2), and extending to (4).
4. Europe during the Renaissance and the Reformation, roughly the fifteenth and sixteenth centuries.
5. The seventeenth and eighteenth centuries.
6. The nineteenth century.
7. The twentieth century.

This division follows loosely the general development of Western civilization and its indebtedness to the Near East. Possibly (6), (7) are only one, although profoundly significant new trends became evident shortly after 1900. In the sequel, we shall observe what appears to have been the main contribution in each of the seven periods. A few anticipatory remarks here may clarify the picture for those seeing it for the first time.

Although the peoples of the Near East were more active than the Europeans during the third of the seven periods, mathematics as it exists today is predominantly a product of Western civilization. Ancient advances in China, for example, either did not enter the general stream or did so by commerc not yet traced. Even such definite techniques as were devised either belong to the trivia of mathematics or were withheld from European mathematicians until long after their demonstrably independent invention in Europe. For-example, Horner's method for the numerical solution of ~~equations may have been known to the Chinese, but Horner did not know that it was. And, as a matter~~ of fact, mathematics would not be much the poorer if neither the Chinese nor Horner had ever hit on the method.

European mathematics followed a course approximately parallel to that of the general culture in th several countries. Thus the narrowly practical civilization of ancient Rome contributed nothing to mathematics; when Italy was great in art, it excelled in algebra; when the last surge of the Elizabetha age in England had spent itself, supremacy in mathematics passed to Switzerland and France. Frequently, however, there were sporadic outbursts of isolated genius in politically minor countries, in the independent creation of non-Euclidean geometry in Hungary in the early nineteenth century. Sudden upsurges of national vitality were occasionally accompanied by increased mathematical activity, as in the Napoleonic wars following the French Revolution, also in Germany after the disturbances of 1848. But the world war of 1914-18 appears to have been a brake on mathematical progress in Europe and to a lesser degree elsewhere, as also were the subsequent manifestations of nationalism in Russia, Germany, and Italy. These events hastened the rapid progress which mathematics had been making since about 1890 in the United States of America, thrusting that country into a leading position.

The correlation between mathematical excellence and brilliance in other aspects of general culture was sometimes negative. Several instances might be given; the most important for the development o mathematics falls in the Middle Ages. When Gothic architecture and Christian civilization were at their zenith in the twelfth century (some would say in the thirteenth), European mathematics was jus beginning the scent from its nadir. It will be extremely interesting to historians eight centuries hence if it shall appear that the official disrepute into which mathematics and impartial science had fallen i certain European countries some years before the triumph, of medieval ideals in September, 1939, wa the dawn of a new faith about to enshrine itself in the unmathematical simplicities of a science-less architecture. Our shaggy ancestors got along for hundreds of thousands of years without science or mathematics in their filthy caves, and there is no obvious reason why our brutalized descendants—if they are to be such—should not do the same.

Attending here only to acquisitions of the very first magnitude in all seven of the periods, we may signalize three. All will be noted in some detail later.

The most enduringly influential contribution to mathematics of all the periods prior to the Renaissance was the Greek invention of strict deductive reasoning. Next in mathematical importance is the Italian and French development of symbolic algebra during the Renaissance. The Hindus of the seventh to the twelfth centuries A.D. had almost invented algebraic symbolism; the Moslems reverte in their classic age to an almost completely rhetorical algebra. The third major advance has already been indicated, but may be emphasized here: in the earlier part of the fifth period—seventeenth century—the three main streams of number, form, and continuity united. This generated the calculus and mathematical analysis in general; it also transformed geometry and made possible the later creation of the higher spaces necessary for modern applied mathematics. The leaders here were French, English, and German.

The fifth period is usually considered as the fountainhead of modern pure mathematics. It brackets the beginning of modern science; and another major advance was the extensive application of the newly created pure mathematics to dynamical astronomy, following the work of Newton, and, a little later, to the physical sciences, following the methodology of Galileo and Newton. Finally, in the nineteenth century, the great river burst its banks, deluging wildernesses where no mathematics had flourished and making them fruitful.

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If the mathematics of the twentieth century differs significantly from that of nineteenth, possibly the most important distinctions are a marked increase in abstractness with a consequent gain in generality, and a growing preoccupation with the morphology and comparative anatomy of mathematical structures; a sharpening of critical insight; and a dawning recognition of the limitation of classical deductive reasoning. If 'limitations' suggests frustration after about seven thousand year of human strivings to think clearly, the suggestion is misleading. But it is true that the critical evaluations of accepted mathematical reasoning which distinguished the first four decades of the twentieth century necessitated extensive revisions of earlier mathematics, and inspired much new work of profound interest for both mathematics and epistemology. They also led to what appeared to be the final abandonment of the theory that mathematics is an image of the Eternal Truth.

The division of mathematical history into about seven periods is more or less traditional and undoubtedly is illuminating, especially in relation to the fluctuating light which we call civilization. But the unorthodox remote, middle, and recent periods, described earlier, seem to give a truer presentation of the development of mathematics itself and a more vivid suggestion of its innate vitality.

## *Some general characteristics*

In each of the seven periods there was a well-defined rise to maturity and a subsequent decline in each of several limited modes of mathematical thought. Without fertilization by creative new ideas, each was doomed to sterility. In the Greek period, for example, synthetic metric geometry, as a method, got as far as seems humanly possible with our present mental equipment. It was revivified into something new by the ideas of analytic geometry in the seventeenth century, by those of projective geometry in the seventeenth and nineteenth centuries, and finally, in the eighteenth and nineteenth centuries, by those of differential geometry.

Such revitalizations were necessary not only for the continued growth of mathematics but also for the development of science. Thus it would be impossible for mathematicians to apprehend the subtle complexities of the geometries applied to modern science by the methods of Euclid and Apollonius. And in pure mathematics, much of the geometry of the nineteenth century was thrust aside by the more vigorous geometries of abstract spaces and the non-Riemannian geometries developed in the twentieth. Considerably less than forty years after the close of the nineteenth century, some of the geometrical masterpieces of that heroic age of geometry were already beginning to seem otiose and antiquated. This appears to be the case for much of classical differential geometry and synthetic projective geometry. If mathematics continues to advance, the new geometries of the twentieth century will likely be displaced in their turn, or be subsumed under still rarer abstractions. In mathematics, of all places, finality is a chimera. Its rare appearances are witnessed only by the mathematically dead.

As a period closes, there is a tendency to overelaboration of merely difficult things which the succeeding period either ignores as unlikely to be of lasting value, or includes as exercises in more powerful methods. Thus a host of special curves investigated with astonishing vigor and enthusiasm by the early masters of analytic geometry live, if at all, only as problems in elementary textbooks. Perhaps the most extensive of all mathematical cemeteries are the treatises which perpetuate artificially difficult problems in mechanics to be worked as if Lagrange, Hamilton, and Jacobi had never lived.

Again, as we approach the present, new provinces of mathematics are more and more rapidly strip~~ped of their superficial riches, leaving only a hypothetical mother lode to be sought by the better~~ equipped prospectors of a later generation. The law of diminishing returns operates here in mathematics as in economics: without the introduction of radically new improvements in method, th income does not balance the outgo. A conspicuous example is the highly developed theory of algebraic invariants, one of the major acquisitions of the nineteenth century ; another, the classical theory of multiply periodic functions, of the same century. The first of these contributed indirectly to the emergence of general relativity; the second inspired much work in analysis and algebraic geometry.

A last phenomenon of the entire development may be noted. At first the mathematical disciplines were not sharply defined. As knowledge increased, individual subjects split off from the parent mass and became autonomous. Later, some were overtaken and reabsorbed in vaster generalizations of the mass from which they had sprung. Thus trigonometry issued from surveying, astronomy, and geometry only to be absorbed, centuries later, in the analysis which had generalized geometry.

This recurrent escape and recapture has inspired some to dream of a final, unified mathematics which shall embrace all. Early in the twentieth century it was believed by some for a time that the desired unification had been achieved in mathematical logic. But mathematics, too irrepressibly creative to be restrained by any formalism, escaped.

## *Motivation in mathematics*

Several items in the foregoing prospectus suggest that much of the impulse behind mathematics ha been economic. In the third and fourth decades of the twentieth century, for obvious political reasons attempts were made to show that all vital mathematics, particularly in applications, is of economic origin.

To overemphasize the immediately practical in the development of mathematics at the expense of sheer intellectual curiosity is to miss at least half the fact. As any moderately competent mathematician whose education has not stopped short with the calculus and its commoner application may verify for himself, it simply is not true that the economic motive has been more frequent than th purely intellectual in the creation of mathematics. This holds for practical mathematics as applied in commerce, including all insurance, science, and the technologies, as well as for those divisions of mathematics which at present are economically valueless. Instances might be multiplied indefinitely four must suffice here, one from the theory of numbers, two from geometry, and one from algebra.

About twenty centuries before the polygonal numbers were generalized, and considerably later applied to insurance and to statistics, in both instances through combinatorial analysis, the former by way of the mathematical theory of probability, their amusing peculiarities were extensively investigated by arithmeticians without the least suspicion that far in the future these numbers were to prove useful in practical affairs. The polygonal numbers appealed to the Pythagoreans of the sixth century B.C. and to their bemused successors on account of the supposedly mystical virtues of such numbers. The impulse here might be called religious. Anyone familiar with the readily available history of these numbers and acquainted with Plato's dialogues can trace for himself the thread of number mysticism from the crude numerology of the Pythagoreans to the Platonic doctrine of Ideas. None of this greatly resembles insurance or statistics.

Later mathematicians, including one of the greatest, regarded these numbers as legitimate objects of intellectual curiosity. Fermat, cofounder with Pascal in the seventeenth century of the mathematic theory of probability, and therefore one of the grandfathers of insurance, amused himself with the polygonal and figurate numbers for years before either he or Pascal ever dreamed of defining probability mathematically.

As a second and somewhat hackneyed instance, the conic sections were substantially exhausted by the Greeks about seventeen centuries before their applications to ballistics and astronomy, and through the latter to navigation, were suspected. These applications might have been made without th Greek geometry, had Descartes' analytics and Newton's dynamics been available. But the fact is that by heavy borrowings from Greek conics the right way was first found. Again the initial motive was intellectual curiosity.

The third instance is that of polydimensional space. In analytic geometry, a plane curve is represented by an equation containing two variables, a surface by an equation containing three. Cayle in 1843 transferred the language of geometry to systems of equations in more than three variables, thus inventing a geometry of any finite number of dimensions. This generalization was suggested directly by the formal algebra of common analytic geometry, and was elaborated for its intrinsic interest before uses for it were found in thermodynamics, statistical mechanics, and other departmen of science, including statistics, both theoretical and industrial, as in applied physical chemistry. In passing, it may be noted that one method in statistical mechanics makes incidental use of the arithmetical theory of partitions, which treats of such problems as determining in how many ways a given positive integer is a sum of positive integers. This theory was initiated by Euler in the eighteenth century, and for over 150 years was nothing but a plaything for experts in the perfectly useless theory of numbers.

The fourth instance concerns abstract algebra as it has developed since 1910. Any modern algebrai may easily verify that much of his work has a main root in one of the most fantastically useless problems ever imagined by curious man, namely, in Fermat's famous assertion of the seventeenth century that $x^n + y^n = z^n$ is impossible in integers $x$, $y$, $z$ all different from zero if $n$ is an integer great than two. Some of this recent algebra quickly found use in the physical sciences, particularly in modern quantum mechanics. It was developed without any suspicion that it might be scientifically useful. Indeed, not one of the algebraists concerned was competent to make any significant applicatio of his work to science, much less to foresee that such applications would some day be possible. As la as the autumn of 1925., only two or three physicists in the entire world had any inkling of the new channel much of physics was to follow in 1926 and the succeeding decade.

## *Residues of epochs*

In following the development of mathematics, or of any science, it is essential to remember that although some particular work may now be buried it is not necessarily dead. Each epoch has left a mass of detailed results, most of which are now of only antiquarian interest. For the remoter periods, these survive as curiosities in specialized histories of mathematics. For the middle and recent period —since the early decades of the seventeenth century—innumerable theorems and even highly developed theories are entombed in the technical journals and transactions of learned societies, and are seldom if ever mentioned even by professionals. The mere existence of many is all but forgotten. The lives of thousands of workers have gone into this moribund literature. In what sense do these hal forgotten things live ? And how can it be truthfully said that the labor of all those toilers was not wasted ?

The answers to these somewhat discouraging questions are obvious to anyone who works in